Research Statement

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My interests are algebraic geometry, representation theory, and geometric invariant theory. My Ph.D. thesis concerns: 1) the relation between conformal blocks and the moduli of G-bundles on a (possibly singular, reducible) curve, and 2) the tensor decomposition problem (how does a tensor product of group representations decompose into irreducibles).

In the first problem, I used conformal blocks to give a compactification of the moduli of G-bundles over Deligne-Mumford's stack of stable curves $\overline{\mathcal{M}}_g$. I showed that the sheaf of conformal blocks algebras is finitely generated and that, at sufficiently high level, conformal blocks can be identified with sections of an ample line bundle on a certain compactified moduli space of G-bundles. This generalizes a result of Belkale-Gibney for $G = \mathrm{SL}(r)$. This is discussed further in §1.

In the second, I studied certain divisors in flag varieties which arise from the tensor decomposition problem. By Borel-Weil theory, the tensor problem is related to describing the G-effective cone of line bundles on the flag variety $(G/B)^n$ that have a nonempty semistable locus. Unstable loci in $(G/B)^n$ can be described using Schubert calculus of Grassmannians G/P, and, in the case $G = \mathrm{SL}(r)$, I gave a simple numerical description of their possible codimension one components. This has interesting consequences for studying GIT quotients of $(\mathrm{SL}(r)/B)^n$. More details are given in §2.

Generalizations and related questions are described briefly in §3.

1. Conformal blocks and G-bundles on curves.

For a simple Lie algebra $\mathfrak{g}$, integer $l \geq 0$, and stable curve C, let $\mathbb{V}_{\mathfrak{g},l,C}$ be the associated space of conformal blocks. These are vector spaces which originated from physics and can be defined in terms of reprentation theory of the affine Lie algebra of $\mathfrak{g}$. Conformal blocks form vector bundles $\mathbb{V}_{\mathfrak{g},l}$ on the stack of genus g stable curves $\overline{\mathcal{M}}_g$, which have been of great interest to algebraic geometers at least since the '90s, when the search for a proof of the Verlinde formula initiated an intense study of conformal blocks by Beauville-Laszlo [2], Faltings [10], Kumar-Narasimhan-Ramanathan [12], Laszlo-Sorger [14], and many others.

Conformal blocks are deeply connected to moduli of G-bundles on C. For example, for a sufficiently high level l and *smooth* curve C of genus $g \geq 2$, $\mathbb{V}^*_{\mathfrak{g},l,C}$ is isomorphic to sections of an ample line bundle on Ramanathan's moduli space of G-bundles $\mathfrak{M}_G(C)$. In particular, $\mathrm{Proj}\,\mathcal{A}(C) \cong \mathfrak{M}_G(C)$, where $\mathcal{A}(C) \coloneqq \bigoplus_{l\geq 0} \mathbb{V}^*_{\mathfrak{g},l,C}$ (see [15] for the algebra structure).

But, the moduli space of G-bundles is no longer compact for a singular curve $C \in \overline{\mathcal{M}}_g$, just as in the vector bundle case where one needs to allow vector bundles to degenerate into torsion-free sheaves to obtain compactifications. While $\mathrm{Proj}\,\mathcal{A}(C)$ is a natural candidate for a compactification, it has no obvious modular interpretation, and we need to show that $\mathcal{A}(C)$ is a finitely generated algebra to ensure that $\mathrm{Proj}\,\mathcal{A}(C)$ is actually a projective variety.

In [19], I prove finite generation of $\mathcal{A}(C)$ for any stable curve $C \in \overline{\mathcal{M}}_g$ and give a modular interpretation of $\mathrm{Proj}\,\mathcal{A}(C)$. Finite generation was previously only known for $\mathfrak{g} = \mathfrak{sl}(r)$ (due to Belkale-Gibney [5] and Moon-Yoo [16]), and no modular interpretation was known. I show that, for sufficiently large m, the subalgebra $\mathcal{A}^m(C) = \bigoplus_{l\geq 0} \mathbb{V}^*_{\mathfrak{g},lm,C}$ is isomorphic to the section ring of an ample line bundle on a certain compactification of the moduli space of G-bundles, called the the moduli space of "singular G-bundles," which was introduced by Schmitt and Muñoz-Castañeda ([18]).

The proof of finite generation of $\mathcal{A}(C)$ for every $C \in \overline{\mathcal{M}}_g$ implies finite generation of the sheaf of algebras $\mathcal{A} = \bigoplus_{l\geq 0} \mathbb{V}^*_{\mathfrak{g},l}$ on $\overline{\mathcal{M}}_g$, and thus $\mathfrak{X} = \operatorname{Proj}\mathcal{A}$ gives a flat, projective family $\mathfrak{X} \to \overline{\mathcal{M}}_g$ whose fiber over a smooth curve is Ramanathan's moduli space of G-bundles and whose fiber over an arbitrary stable curve is a moduli space of singular G-bundles. This interpretation only applies to closed fibers of $\mathfrak{X}$ over $\overline{\mathcal{M}}_g$, and it still remains to find a modular interpretation of $\mathfrak{X}$ that works for the relative setting (if possible). There are many other possible approaches, for example Balaji has constructed, using Bruhat-Tits theory, flat moduli spaces over $\operatorname{Spec}\mathbb{C}[[t]]$ that give degenerations of the moduli of G-bundles in the setting of a smooth curve degenerating into an irreducible curve with one node [1]. It would be interesting to see how Balaji's moduli spaces relate to the conformal blocks compactification.

2. Tensor decomposition problem.

Let G be a connected semi-simple algebraic group over $\mathbb{C}$. The tensor decomposition problem asks, "Given two irreducible representations V_λ and V_μ of G, which irreducible representations V_ν appear in $V_\lambda \otimes V_\mu$?" By the Borel-Weil theorem, a triple of dominant integral weights (λ, μ, ν) satisfies this condition if and only if a certain line bundle on $(G/B)^3$ has a nonzero G-invariant section, where $B \subset G$ is a Borel subgroup. Thus, if the problem is loosened to looking for triples (λ, μ, ν) such that $V_{n\nu} \subseteq V_{n\lambda} \otimes V_{n\mu}$ for some $n > 0$ (this is called the "saturated tensor problem"), then the tensor problem becomes equivalent to the GIT problem of describing the G-effective cone $C \subseteq \operatorname{Pic}(G/B)^3$ of line bundles with a nonempty semistable locus (here G is acting diagonally on $(G/B)^3$). For more detailed background on the tensor problem, see Kumar's article [13].

In [3], Belkale-Kumar gave a set of inequalities defining the cone C, and this was proved to be a minimal set of inequalities by Ressayre in [17]. In [20], I generalize this result by finding inequalities for the sequence of subcones

$$C_k = \{\mathcal{L} \in \operatorname{Pic}(G/B)^n : \text{unstable locus of } \mathcal{L} \text{ has codimension} \geq k\}.$$

The proof uses a natural relationship between the Hesselink stratification of $(G/B)^n$ and intersection theory of Grassmannians G/P. The inequalities are indexed by Schubert positions $v_1, \ldots, v_n \in W$ (Weyl group of G) for G/P, ranging over all maximal parabolics P. Following Belkale-Kumar, I showed that one only needs to check the inequalities indexed by "Levi-movable" Schubert positions, which are Schubert positions such that translates of Schubert varieties by general elements of the Levi subgroup meet transversally at $eP \in G/P$ (if the intersection has negative expected dimension – in which case "transversality" isn't really defined – then we just require that the tangent space of the intersection at eP is zero for general Levi translates). In the case $k = 1, 2$ (expected dimension zero or -1), this Levi movability condition can be tested using Belkale-Kumar's deformed product of cohomology ([3]).

The extremal rays of the cone C were determined in [6]. A large class of extremal rays, "F-rays," arise from GIT in the following way. Let $D \subset (G/B)^n$ be a G-invariant irreducible divisor such that $\dim \mathrm{H}^0(\mathcal{O}(mD))^G = 1$ for all $m \geq 0$. Then $\mathcal{O}(D)$ spans an extremal ray called an F-ray, and D is called an F-divisor. A (reduced, irreducible) divisor is an F-divisor if and only if it is a component of the unstable locus of some line bundle on $(G/B)^n$, and the possible F-divisors which may appear in an unstable locus can be checked by Belkale-Kumar inequalities.

For $G = \mathrm{SL}(r)$, I give a description of all F-divisors as intersection loci

$$N_{\vec{v}} = \{(g_1 B, \dots, g_n B) \in (\mathrm{SL}(r)/B)^n : \bigcap_{i=1}^{n} g_i X_{v_i} \neq \varnothing\}$$

satisfying a simple numerical condition, where $X_v \subset \mathrm{SL}(r)/P$ is the Schubert variety corresponding to $v \in W$. As a consequence of the numerical condition, I show that one can always assume a line bundle on $X = (\mathrm{SL}(r)/B)^n$ has unstable locus of codimension ≥ 2 for the purposes of moduli problems. In particular there is a well-defined injective map

$$\mathrm{Pic}(X /\!\!/ G) \hookrightarrow \mathrm{Pic}\, X$$

for any GIT quotient of X. More can be said about the image of this map using Kempf's descent lemma as in [8], and it would be interesting to use this to further study the Picard groups of GIT quotients of $(G/B)^n$.

3. Future work

3.1 Conformal blocks and G-bundles on curves

(i) Prove finite generation and explore modular interpretations for conformal blocks algebras on the stack of n-pointed curves $\overline{\mathcal{M}}_{g,n}$. Finite generation was proven for $\mathfrak{g} = \mathfrak{sl}(r)$ by Moon-Yoo ([16]), and one would like to generalize this to arbitrary simple Lie algebras.

(ii) Spaces of conformal blocks are isomorphic to sections of determinant of cohomology line bundles on the stack of G-bundles $\mathrm{Bun}_G(C)$ (or possibly fractional powers of determinant of cohomology). Replacing curves with an appropriate class of higher dimensional varieties Z, is the section ring of determinant of cohomology on $\mathrm{Bun}_G(Z)$ still finitely generated?

(iii) Recently it has been shown that certain modules over vertex operator algebras give rise to generalized sheaves of conformal blocks on $\overline{\mathcal{M}}_{g,n}$ (see for example [9]). How can one define an algebra structure on the sum of these sheaves, and when does this algebra have nice properties like finite generation or normality? Do these sheaves have a geometric interpretation that generalizes the relationship between traditional conformal blocks and moduli of G-bundles?

3.1 Tensor decomposition problem

(i) First I would like to extend my results on unstable divisors in $(\mathrm{SL}(r)/B)^n$ to arbitrary semisimple groups, if possible. Then there are several further generalizations, which I will describe in (ii), (iii).

(ii) The first generalization is the "branching problem," where one studies restrictions of representations for an embedding of groups $G \subset \widehat{G}$ (the tensor problem corresponds to the diagonal embedding $G \subset G^n$). The associated GIT problem takes place in the G-variety $\widehat{G}/\widehat{B}$ for a Borel subgroup $\widehat{B} \subset \widehat{G}$, and this version has been studied extensively by Ressayre, Kiers, and others. Inequalities and rays for the associated cone were determined in [11], [17]. The unstable loci are again controlled by Schubert calculus of Grassmannians, so it would be interesting to study their divisorial components more closely.

(iii) There is also a "quantum version" of the tensor problem, where the flag variety $(G/B)^n$ is replaced by the stack of parabolic G-bundles on a curve, and one wants to describe the cone of effective line bundles on this stack. The cone's defining inequalities were determined by Belkale-Kumar in [4], and its rays were only recently determined for $G = \mathrm{SL}(r)$ by Belkale [7]. Finding the rays for arbitrary groups remains open. This version of the problem also has a rich connection to rigid local systems (see [7]).

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